FACTORIZATION OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES.

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The theorems of factorization with which this paper deals are in part so obvious as to seem trivial. In part, however, they involve definitions and conceptions, the necessity for which is not evident at a glance, nor can their justification be established on any *a priori* grounds; one has to go over the whole field and ascertain what the situation is that is to be met.

The main theorems are contained in a memoir by Weierstrass (cf. § 3 below); but the presentation in that paper is altogether inadequate for making clear to the reader that which is essential.

This theory of factorization is of much importance in practice, and it is embarrassing for a writer, in studying a subject in the theory of analytic functions of several complex variables, to be obliged to assume it, well knowing that most readers have never thought it through in detail. It is for this reason that I have ventured to offer a systematic presentation of the theory as a whole.*

The reader will do well to turn at once to § 4, for he will find there one of the chief topics with which the paper deals.

Contents.

§ 1. Cauchy's Contribution.

Definitions. A function $F(z_1, \cdots, z_n)$ of several complex variables is said to be *analytic in* or *at* a point $(z_1, \cdots, z_n) = (a_1, \cdots, a_n)$—or more briefly $(z) = (a)$—if it is possible to embed each coördinate a_k in a region T_k of the complex z_k-plane, these regions being chosen in such a manner that

* A paper by Dautheville, Annales de l'École Normale Supérieure, (3) 2 (1885) supplement, is also devoted to an exposition of this subject.

(a) F is defined at each point (z), where z_k is an arbitrary point of T_k, $k = 1, \cdots, n$;

(b) the n partial derivatives

$$\frac{\partial F}{\partial z_k}, \qquad k = 1, \cdots, n,$$

all exist at every such point;

(c) these derivatives are all continuous at all the points in question.

It has been shown that condition (c) is a consequence of conditions (a) and (b); but the proof is not elementary, and does not belong at the beginning of the theory.

If F is analytic at the point $(z) = (a)$, then F can be developed by Taylor's Theorem about the point (a).

The domain of the points $(z_1, \cdots, z_n)$ in which a function F is considered may be any $2n$-dimensional region T of the real space of the $2n$ real variables $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ where

$$z_k = x_k + iy_k.$$

The function F is said to be *analytic in* T if it is analytic at each point of T.

A special class of regions T of use in practice consists of the *cylindrical regions*, i. e., regions such that each z_k is, as in the above definition, an arbitrary point of a fixed region T_k of the z_k-plane, $k = 1, \cdots, n$.

Cauchy's Analysis. Let $F(w, z)$ be a function of the two complex variables w, z, which is analytic in the point (b, a), and let

$$F(b, a) = 0, \qquad F(w, a) \not\equiv 0.$$

Then $F(w, a)$ will have a root of the mth order in $w = b$, where $m \geqq 1$; and it will have no second root in the neighborhood of the point $w = b$.

Let

$$|w - b| < r_1, \qquad |z - a| < h_1$$

be a region in which F is analytic. Then it is possible to choose a positive $r < r_1$ so that $F(w, a)$ does not vanish at any point of the region $0 < |w - b| \leqq r$:

$$F(w, a) \neq 0, \qquad 0 < |w - b| \leqq r.$$

Let W be an arbitrary point of the circle

$$|W - b| = r.$$

We can then find a positive $h \leqq h_1$ such that $F(W, z)$ will not vanish for any z of the circle $|z - a| < h$:

$$F(W, z) \neq 0, \qquad \text{if} \qquad |W - b| = r, \qquad |z - a| < h.$$

For, the points (w, z) of the four-dimensional space of these variables, in which $w = W$ and $z = a$, i. e., the points

$$|W - b| = r, \qquad z = a,$$

form a real curve or *chain*—more specifically a *connected perfect manifold*—in each point of which $F(w, z)$ is continuous and different from 0. It must, therefore, be possible to embed this manifold in a four-dimensional neighborhood, at each point (w, z) of which $F(w, z)$ is different from 0, and clearly this neighborhood can be chosen as cylindrical. If T_2 is the corresponding region of the z-plane, then h can be taken as the shortest distance from $z = a$ to a boundary point of T_2—or as any smaller positive quantity. For convenience, we will henceforth take $b = 0$.

And now I say: *If z be an arbitrary point of the region*

$$S: \qquad |z - a| < h,$$

*then $F(w, z)$, regarded as a function of w alone, has precisely m roots in the region**

$$K: \qquad |w| < r.$$

In fact, for such a value of z, $F(w, z)$ has no root on the circle

$$C: \qquad |W| = r,$$

and the number of roots within the circle is given by the formula

$$\frac{1}{2\pi i}\int_C \frac{F_w(W, z)dW}{F(W, z)}, \qquad F_w(w, z) = \frac{\partial F}{\partial w},$$

where the integral is extended in the positive sense over C. In the point $z = a$ this integral has the value m. Moreover, it is a continuous function of z, and being an integer, it must, therefore, remain constant.

Let $\varphi(w)$ be any function of w which is analytic within C and continuous on the boundary. Then the integral

$$\frac{1}{2\pi i}\int_C \varphi(W)\frac{F_w(W, z)dW}{F(W, z)},$$

extended in the positive sense over the circle C, has for its value†

$$\varphi(w_1) + \cdots + \varphi(w_m) = P(z),$$

where $w_1, \cdots, w_m$ are the m roots of $F(w, z)$. The function $P(z)$ is seen from inspection of the integral to be analytic within C.

* Cauchy, Lithographed Turin memoir of October 11, 1831 = Execices d'analyse, vol. 2 (1841), p. 64.

† Cauchy, l. c., p. 66.

If, in particular, $m = 1$ and we set $\varphi(w) = w$, the above equation becomes

$$w = P(z),$$

and thus the roots w of $F(w, z)$ which lie in the circle C when z is a point of S form a function which is analytic in S. We have here, probably, the first rigorous proof of an existence theorem for an implicit function in a general case.*

So far Cauchy carried his analysis in the year 1831. We turn now to the further development due to Weierstrass in the sixties.

§ 2. A General Theorem on Implicit Functions.

If, in the preceding paragraph, we set† $\varphi(w) = w^k$, we find:

$$w_1^k + \cdots + w_m^k = P_k(z),$$

where $P_k(z)$ is analytic in the circle S. Let

$$\begin{aligned} A_1 &= -(w_1 + \cdots + w_m), \\ A_2 &= w_1 w_2 + w_1 w_3 + \cdots + w_{m-1} w_m, \\ &\cdots\cdots\cdots\cdots \\ A_m &= (-1)^m w_1 \cdots w_m. \end{aligned}$$

Then A_k can be expressed as a polynomial in $P_1(z), \cdots, P_m(z)$, and hence is a function of z analytic in S.

The foregoing analysis applies with mere formal modifications to the case of a function $F(w, z_1, \cdots, z_n)$, and thus we obtain the following

THEOREM. *Let $F(w, z_1, \cdots, z_n)$ be analytic in the point $(b, a_1, \cdots, a_n)$, and let*

$$F(b, a_1, \cdots, a_n) = 0, \qquad F(w, a_1, \cdots, a_n) \not\equiv 0.$$

For simplicity, let $b = 0$.

There exists then a definite neighborhood K of the point $w = 0$ and a neighborhood S of the point $(a_1, \cdots, a_n)$,

$$K: \qquad |w| < r;$$

$$S: \qquad |z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$$

such that, if (z) be an arbitrary point of S, the equation

$$F(w, z_1, \cdots, z_n) = 0$$

has precisely m roots in K.

* Cauchy, l. c., p. 71.

† The form of the proof of Weierstrass's Theorem of § 3 which begins with this substitution is due to Simart. Cf. Picard, Traité d'analyse, vol. 2, ch. 9.

Moreover, these roots are given by an equation of the form

$$w^m + A_1 w^{m-1} + \cdots + A_m = 0,$$

where $A_j(z_1, \cdots, z_n)$, $j = 1, \cdots, m$, *is analytic in* S *and vanishes at* $(a_1, \cdots, a_n)$.

This theorem is due to Weierstrass, who deduced it from his theorem of factorization, to which we turn in § 3. But the proof is essentially simpler than the one he gave and rests, as the reader perceives, on material already furnished by Cauchy in that remarkable memoir which contained the proof of Taylor's theorem for analytic functions of a complex variable.

This theorem is used in practice much oftener than the general theorem of factorization, and it is well to have a simple proof of it.

§ 3. Weierstrass's Theorem of Factorization.

Let

$$f(w, z) = w^m + A_1 w^{m-1} + \cdots + A_m,$$

and form the function

$$\frac{F(w, z)}{f(w, z)}.$$

Assign to z an arbitrary value in S and hold it fast. Then the above expression becomes a function of w alone:

$$\frac{F(w, z)}{f(w, z)} = \frac{F(w, z)}{(w - w_1) \cdots (w - w_m)} = \Phi(w).$$

This function is analytic within K except for removable singularities in each of the points w_k, the numerator and the denominator having roots of the same order there. If it is defined in these points as equal to its limiting value, it then becomes a function analytic without exception throughout the interior of K and nowhere zero there.

Moreover, $\Phi(w)$ is continuous on the boundary of K. Hence it can be represented by Cauchy's integral formula:

$$\Phi(w) = \frac{1}{2\pi i} \int_C \frac{F(W, z)}{f(W, z)} \cdot \frac{dW}{W - w}.$$

The expression on the right of this equation represents, however, a function $\Omega(w, z)$ of the two independent variables w, z analytic throughout the cylindrical region

$$|w| < r, \qquad |z - a| < h;$$

and we have already observed that it is never zero there.

We have, then, the relation

$$F(w, z) = f(w, z)\Omega(w, z),$$

and this equation is Weierstrass's Theorem of Factorization.

The above analysis holds with only formal modifications for the case of a function $F(w, z_1, \cdots, z_n)$, and hence we may state the result as follows.

WEIERSTRASS'S THEOREM OF FACTORIZATION. *Let $F(w, z_1, \cdots, z_n)$ be analytic in the point $(b, a_1, \cdots, a_n)$, and let*

$$F(b, a_1, \cdots, a_n) = 0, \qquad F(w, a_1, \cdots, a_n) \not\equiv 0.$$

For simplicity, let $b = 0$.

Then F can be represented throughout a certain neighborhood of the point $(0, a_1, \cdots, a_n)$,

$$\mathfrak{T}: \qquad |w| < r; \qquad |z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$$

in the form

$$F(w, z_1, \cdots, z_n) = (w^m + A_1 w^{m-1} + \cdots + A_m)\Omega(w, z_1, \cdots, z_n),$$

where $A_k = A_k(z_1, \cdots, z_n)$, $k = 1, \cdots, m$, denotes a function analytic in the region

$$S: \qquad |z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$$

and vanishing in the point $(a_1, \cdots, a_n)$; and Ω is analytic in $\mathfrak{T}$ and does not vanish there.

Weierstrass tells us that he gave this theorem repeatedly in his university lectures at Berlin from 1860 on. He published it in lithographed form in the year 1879, and it was printed in the Funktionenlehre of 1886, p. 107; cf. Werke, 2, p. 135. Poincaré gave a proof of the theorem in his Thèse of 1879.

A number of new proofs of the theorem have been given in recent years. Cf. Bliss, Princeton Colloquium, p. 49, and the Transactions of the Amer. Math. Soc., 13 (1912), p. 133, where further literature is cited.

The Case of Two Variables. In the case $n = 1$ a more general theorem can be stated.

THEOREM. *Let $F(w, z)$ be analytic in the point (b, a), and let*

$$F(b, a) = 0, \qquad F(w, z) \not\equiv 0.$$

For simplicity let $b = 0$.

Then $F(w, z)$ can be represented in the neighborhood of the point $(0, a)$ in the form

$$F(w, z) = z^\lambda f(w, z)\Omega(w, z),$$

where $f(w, z) \equiv 1$ *or else*

$$f(w, z) = w^m + A_1 w^{m-1} + \cdots + A_m,$$

$A_k = A_k(z)$ *being analytic in the point* a *and vanishing there; and* $\Omega(w, z)$ *being analytic in the point* $(0, a)$ *and not vanishing there.*

This theorem was used by Black* and it forms an essential element in the study of envelopes by Risley and MacDonald.† It does not admit the generalization for $n > 1$ which suggests itself.‡

Linear Transformation. Let $\Phi(x_0, x_1, \cdots, x_n)$ be a function of $n + 1$ complex variables analytic in the point $(a_0, a_1, \cdots, a_n)$ and vanishing there, but not vanishing identically. It may happen that

$$\Phi(x_0, a_1, \cdots, a_n) \equiv 0,$$

and that the same is true if any other x_k be singled out. It is then impossible to apply the theorem of factorization to the function Φ as it stands.

By means, however, of a suitable linear transformation,

$$\begin{aligned} w &= c_{00}x_0 + c_{01}x_1 + \cdots + c_{0n}x_n + c_0, \\ z_1 &= c_{10}x_0 + c_{11}x_1 + \cdots + c_{1n}x_n + c_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_n &= c_{n0}x_0 + c_{n1}x_1 + \cdots + c_{nn}x_n + c_n, \end{aligned}$$

it is possible to carry Φ over into a function F:

$$\Phi(x_0, x_1, \cdots, x_n) = F(w, z_1, \cdots, z_n)$$

which does satisfy all the conditions of the theorem of factorization, and hence to factor the latter function.§

§ 4. Factors im Kleinen.

Division. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ both be analytic at the point $(a_1, \cdots, a_n)$ and let

$$\Phi(z_1, \cdots, z_n) \not\equiv 0.$$

If there exists a function $Q(z_1, \cdots, z_n)$, also analytic at this point, for which the relation

$$F(z_1, \cdots, z_n) = Q(z_1, \cdots, z_n)\Phi(z_1, \cdots, z_n)$$

* Proceedings Amer. Acad. Arts and Sci., vol. 37 (1902), p. 325.

† Annals of Math., (2) 12 (1911), p. 73.

‡ The Madison Colloquium, Lecture 3, § 2, and Trans. Am. Math. Soc., 17 (1916), p. 1.

§ Weierstrass, l. c.

holds throughout the neighborhood of the point $(a_1, \cdots, a_n)$, then F is said to be *divisible by* Φ *at* or *in the point* $(a_1, \cdots, a_n)$. The function $Q(z_1, \cdots, z_n)$ is called the *quotient* of F by Φ in $(a_1, \cdots, a_n)$, and Φ is called a *factor* of F in or at (a). At those points of the neighborhood of $(a_1, \cdots, a_n)$ in which Φ does not vanish, Q is the arithmetic quotient of F by Φ. But Q is also defined in the points, if any are present, in which $\Phi = 0$, by the method of continuity.

It is to be noted that division by a function which vanishes identically is excluded by the definition.

If F is divisible by Φ at a point (a), then F is divisible by Φ at every other point of a certain neighborhood of (a).

Reducible and Irreducible Functions; Prime Factors. If $F(z_1, \cdots, z_n)$ can be written, in the neighborhood of the point (a), in the form:

$$F(z_1, \cdots, z_n) = F_1(z_1, \cdots, z_n)F_2(z_1, \cdots, z_n),$$

where F_1 and F_2 are both analytic in (a) and vanish there, but neither vanishes identically, then F is said to be *reducible in* or *at the point* (a).

If, on the other hand, $F(z_1, \cdots, z_n)$ is analytic at the point (a) and vanishes there, without however vanishing identically, and if no factorization of the above form is possible, then F is said to be *irreducible in* or *at the point* (a).

The above definitions are seen to relate only to functions $F(z_1, \cdots, z_n)$ which are analytic in a point (a) and vanish there, but do not vanish identically. Other functions $F(z_1, \cdots, z_n)$, which are analytic in the point (a), namely, those which do not vanish there, and the constant 0, are neither reducible nor irreducible in (a).

It does not follow, because a function is irreducible at a point (a), that it is also irreducible at others of its roots which lie near (a). Thus

$$F(x, y, z) = z^2 - x^2y$$

is readily shown to be irreducible at the origin, $(x, y, z) = (0, 0, 0)$. But it is reducible at every root $(0, h, 0)$ for which $h \neq 0$; for

$$F(x, y, z) = (z - x\sqrt{y})(z + x\sqrt{y}).$$

On the other hand, this function is irreducible at the points $(a, 0, 0)$ for all values of a.

An irreducible factor of a function at a given point is called a *prime factor* of the function *at that point.*

Equivalent Functions. Two functions, $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$, are said to be *equivalent* to each other *in* or *at the point* (a) if each is divisible by the other there.

In order that F be equivalent to Φ in (a) it is necessary and sufficient that F be divisible by Φ at (a):

$$F(z_1, \cdots, z_n) = Q(z_1, \cdots, z_n)\Phi(z_1, \cdots, z_n),$$

and furthermore that Q shall not vanish at (a).

If two functions are equivalent to each other at a point, they are also equivalent at every other point of a certain neighborhood of the given point.

We turn next to the statement of one of the most important theorems with which this paper deals.

FUNDAMENTAL THEOREM. *If $F(z_1, \cdots, z_n)$ is analytic in the point $(a_1, \cdots, a_n)$ and vanishes there, but does not vanish identically, then F can be represented in one, and essentially in only one, way as the product of prime factors.*

By *essentially* is meant that, if F be represented by a second product of prime factors, each prime factor of the second representation will be equivalent to a prime factor of the first representation, and the multiplicities of corresponding prime factors in the two representations will be respectively equal.

For the case $n = 1$ the proof of the theorem is immediate. The proof for $n > 1$ is given in § 8.

If the functions F and Φ are analytic in the point (a) and vanish there, and if they have no common divisor which vanishes there, they are said to be *relatively prime in the point* (a), or to have *no common divisor in* (a). And likewise, if F and Φ are both divisible in (a) by a function Ψ which vanishes there, they are said to have a *common divisor in* (a), namely Ψ.

If two functions are relatively prime in a point (a), they are also relatively prime in every other common root of a certain neighborhood of (a).*

The proof of this theorem will be given at the close of § 9.

Greatest Common Divisor. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ be two functions which are analytic in a point (a), and let neither vanish identically. The product of all the irreducible common factors of F and Φ in (a), each taken with the lower of the two multiplicities with which it occurs in F and Φ, is called the *greatest common divisor of F and Φ in* (a).

The roots of this latter function in the neighborhood of (a) yield all the points of that neighborhood in which F and Φ have a common factor, and this function is also the greatest common divisor of F and Φ in each of these points. The proof of this theorem follows at once from the theorem of Weierstrass's just cited.

* Weierstrass, Werke, 2, p. 154.

The least common multiple of F and Φ in a point (a) is defined in a similar manner.

Invariance of the Foregoing Definitions and Theorems. It is obvious that, if the variables $z_1, \cdots, z_n$ be subjected to a non-singular linear transformation,

$$z_i' = c_{i1}z_1 + \cdots + c_{in}z_n + c_i, \qquad i = 1, \cdots, n,$$

a function $F(z_1, \cdots, z_n)$ or a pair of functions F and Φ which satisfy a given one of the above definitions or theorems in a point (a) will go over into a function $F'(z_1', \cdots, z_n')$ or a pair of functions F', Φ' satisfying the same condition in (a').

More generally, the same remark is true if instead of a linear transformation, an arbitrary non-singular analytic transformation is made:

$$z_i' = \varphi_i(z_1, \cdots, z_n), \qquad i = 1, \cdots, n,$$

where, then, each of the functions φ_i is analytic in the point (a) and their Jacobian,

$$\frac{\partial(\varphi_1, \cdots, \varphi_n)}{\partial(z_1, \cdots, z_n)}$$

does not vanish there.

The proof of the fundamental theorem rests on the theory of algebroid polynomials—a class of functions also having independent importance.

§ 5. Algebroid Polynomials.*

By an *algebroid polynomial* is meant a function

$$(A) \qquad f(x, y) = f(x, y_1, \cdots, y_n) = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$

where $a_k = a_k(y_1, \cdots, y_n)$, $k = 0, \cdots, m$, is analytic in the point $(y) = (c)$. For simplicity, we shall usually take the point (c) at the origin, $(c) = (0)$.

If all the coefficients a vanish identically, then f vanishes identically; and conversely. If f does not vanish identically, let

$$a_0(y_1, \cdots, y_n) \not\equiv 0.$$

Then m is defined as the *degree* of f. An algebroid polynomial of degree 0 is any function of $(y_1, \cdots, y_n)$ which is analytic in the origin and does not vanish identically.

We know the fundamental theorem of § 4 to hold for one variable,

* The theory here set forth is closely allied to the theory of ordinary polynomials, and a knowledge of this theory, as given for example in Bôcher's Algebra, Ch. 16, is presupposed on the part of the reader. For the latter's convenience we have changed the notation to conform in the main with that of the reference.

and we assume it to hold for $2, \cdots, n$ variables. By means of the theory developed in this paragraph we shall be able to show that it holds for $n + 1$ variables, and thus establish it generally.

LEMMA 1. *If the algebroid polynomial* (A) *is divisible at the origin by a function* $\psi(y) = \psi(y_1, \cdots, y_n)$, *then every coefficient of* f *is divisible by* $\psi(y)$ *in the origin.*

Since division is subject to the definition of § 4, we have by hypothesis

$$f(x, y) = \omega(x, y)\psi(y), \tag{1}$$

where ω is analytic in the $n + 1$ variables $(x, y_1, \cdots, y_n)$ at the origin, but not by hypothesis is ω an algebroid polynomial. On developing ω according to powers of x:

$$\omega(x, y) = c_0 + c_1 x + c_2 x^2 + \cdots,$$

substituting this series in the right-hand member of (1), and comparing coefficients, the truth of the lemma becomes apparent.

An algebroid polynomial $f(x, y)$ is said to be *primitive* if it is not divisible by any function $\psi(y)$ analytic in the origin and vanishing there. If in particular a primitive algebroid polynomial is of degree 0, it cannot vanish in the origin; and conversely, any function $\psi(y)$ analytic in the origin and not vanishing there is a primitive algebroid polynomial of degree 0.

COROLLARY. *If* $f(x, y)$ *is an algebroid polynomial which does not vanish identically, it can be written as the product of two algebroid polynomials,*

$$f(x, y) = \psi(y)g(x, y),$$

of which the first is of degree 0 *and the second is primitive.*

If, namely, $f(x, y)$ is itself primitive, then $\psi(y)$ can be set $= 1$. If not, $\psi(y)$ can be taken as the greatest common divisor of the coefficients of f.

LEMMA 2. *If the product of two algebroid polynomials,* $f(x, y)$ *and* $\varphi(x, y)$, *is divisible in the origin by a function* $\psi(y)$ *irreducible in the origin, then one of the two factors is divisible by* $\psi(y)$ *in the origin.*

The proof is the same as in the case of ordinary polynomials; cf. Bôcher, l. c., § 73, Theorem 2.

Division. Let $f(x, y)$ and $\varphi(x, y)$ be two algebroid polynomials, of which the second does not vanish identically:

$$\varphi(x, y) = b_0 x^p + b_1 x^{p-1} + \cdots + b_p,$$

$$b_0(y_1, \cdots, y_n) \not\equiv 0, \qquad 0 \leqq p.$$

Then there exists between f and φ a relation of the following form:

$$P(y)f(x, y) = Q(x, y)\varphi(x, y) + R(x, y), \tag{2}$$

where $P(y)$, $Q(x, y)$, $R(x, y)$ are algebroid polynomials, of which the first is of degree 0 and the last, $R(x, y)$, either vanishes identically or is of lower degree than $\varphi(x, y)$. Moreover, P, Q, R shall admit no common factor in the origin.

The proof is the same as for polynomials; Bôcher, l. c., § 63. The relation (2) was given by Weierstrass; cf. Werke, 2, p. 144.

Let

$$P_1(y)f(x, y) = Q_1(x, y)\varphi(x, y) + R_1(x, y)$$

be a second relation satisfying the same conditions. Then $P(y)$ and $P_1(y)$ are equivalent to each other in the origin. Moreover, $R(x, y)$ and $R_1(x, y)$ either both vanish identically or else are equivalent to each other in the origin; and the same is true of $Q(x, y)$ and $Q_1(x, y)$.

To prove this, consider the relation obtained at once from the two given equations:

$$0 = [P_1(y)Q(x, y) - P(y)Q_1(x, y)]\varphi(x, y) \\ + [P_1(y)R(x, y) - P(y)R_1(x, y)].$$

The second [ ] vanishes identically; for otherwise, if $\varphi(x, y)$ be of positive degree, it would be of lower degree than $\varphi(x, y)$, and would be divisible by $\varphi(x, y)$. If φ is of degree 0, then both R and R_1 vanish identically.

It follows, then, that the first [ ] must also vanish identically.

$P(y)$ and $P_1(y)$ are now shown to be equivalent in the origin. For otherwise one of these functions, as $P(y)$, must admit a factor $\psi(y)$ irreducible in the origin which does not divide the other function. From the vanishing of both brackets we infer that $\psi(y)$ divides both $Q(x, y)$ and $R(x, y)$, and this is contrary to hypothesis. The remainder of the proof is now obvious.

Algebraic and Analytic Division. From the analogy with ordinary polynomials it is natural to define f as divisible by φ when, in the relation (2), $R(x, y)$ vanishes identically and $P(y)$ does not vanish, so that it can be set equal to unity:

$$f(x, y) = Q(x, y)\varphi(x, y), \tag{3}$$

where $Q(x, y)$ is an algebroid polynomial. When this condition is fulfilled, we shall say that f is *algebraically divisible by* φ *in the point* $(y) = (0)$, or more generally, $(y) = (c)$.

On the other hand, we already have a definition of divisibility in a point, according to which a relation of the same form, (3), must hold; but here $Q(x, y)$ need not be an algebroid polynomial—it is sufficient that Q be analytic in the point $(x, y_1, \cdots, y_n) = (0, 0, \cdots, 0)$ or $(c, c_1, \cdots, c_n)$.

In distinction from algebraic divisibility just defined we will describe the earlier case now by saying that $f(x, y)$ is *analytically divisible* by $\varphi(x, y)$ in the point $(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n)$.

When f is algebraically divisible by φ in the point $(y) = (c)$, it is obviously analytically divisible by φ in any point

$$(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n), \qquad |c| < \infty.$$

But if f is analytically divisible by φ in a point $(c, c_1, \cdots, c_n)$, it does not follow that f is algebraically divisible by φ in the point $(y) = (c)$. For example, let

$$f(x, y) = x + y,$$

$$\varphi(x, y) = x^2 + (y + 1)x + y = (x + y)(x + 1).$$

Then f is analytically divisible by φ in the origin,

$$Q(x, y) = \frac{1}{x + 1} = 1 - x + x^2 - \cdots.$$

But f is not algebraically divisible by φ in the point $y = 0$.

We can, however, state the following criterion for the coincidence of the two definitions.

THEOREM 1. *In order that the algebroid polynomial $f(x, y)$ be algebraically divisible by the algebroid polynomial $\varphi(x, y)$ in the point*

$$(y_1, \cdots, y_n) = (c_1, \cdots, c_n)$$

it is necessary and sufficient that $f(x, y)$ be analytically divisible by $\varphi(x, y)$ in every point of a certain region

$$\mathfrak{T}: \qquad |x| < \infty, \qquad |y_k - c_k| < h, \qquad k = 1, \cdots, n,$$

*where h is a positive constant, no matter how small.**

The condition is obviously necessary. To prove it sufficient, form the function

$$\frac{f(x, y)}{\varphi(x, y)} = \Omega(x, y).$$

Then Ω has only removable singularities in $\mathfrak{T}$, and shall be defined in these points by its limiting value there. Hence Ω can be developed into a series of the form

$$\Omega(x, y) = c_0 + c_1 x + c_2 x^2 + \cdots,$$

* It would not be sufficient if f were analytically divisible by φ merely in the points

$$(x, y_1, \cdots, y_n) = (c, 0, \cdots, 0),$$

where c is arbitrary, as the example

$$f(x, y) = x, \qquad \varphi(x, y) = yx^2 - x$$

shows.

where each c_k is a function of (y) analytic throughout the region

$$S: \qquad |y_k| < h, \qquad k = 1, \cdots, n,$$

the series converging for all values of x.

Let (y') be a point of S in which the coefficient of the highest power of x in φ is not 0:

$$b_0(y_1', \cdots, y_n') \neq 0,$$

and let σ be a neighborhood of (y') lying wholly within S and such that

$$\eta < |b_0(y_1, \cdots, y_n)|$$

for every point (y) of σ, η being a suitable positive constant. Then, when (y) lies in σ, the roots of $\varphi(x, y)$ remain finite. Hence, if $m \geqq p$, the function

$$\frac{f(x, y)}{x^{m-p}\varphi(x, y)} = \frac{c_0}{x^{m-p}} + \cdots + c_{m-p} + c_{m-p+1}x + \cdots$$

remains finite when (y) lies in σ and $|x| > G$, where G denotes a suitable positive number. It follows, then, that

$$c_k(y_1, \cdots, y_n) \equiv 0, \qquad m - p < k.$$

If $m < p$, $\Omega(x, y)$ would have to vanish identically, and thus this case is impossible. Finally, if $f(x, y)$ vanishes identically, the theorem is granted.

Algebroid Polynomials of Class A. An important class of algebroid polynomials

$$f(x, y) = a_0(x - c)^m + a_1(x - c)^{m-1} + \cdots + a_m,$$

where $a_k = a_k(y_1, \cdots, y_n)$ is analytic in the point $(c_1, \cdots, c_n)$, consists of those for which $m > 0$ and

$$a_0(c_1, \cdots, c_n) \neq 0, \qquad a_k(c_1, \cdots, c_n) = 0, \qquad k = 1, \cdots, m.$$

These shall be denoted as *algebroid polynomials of Class A*. The point $(c, c_1, \cdots, c_n)$ shall be called the *vertex* of f.

An algebroid polynomial $f(x, y)$ of Class A is necessarily primitive. Furthermore, to an arbitrarily small positive ϵ corresponds a positive δ such that, if (y) lies in the region

$$|y_k - c_k| < \delta, \qquad k = 1, \cdots, n,$$

then each root of $f(x, y)$ lies in the circle

$$|x - c| < \epsilon.$$

THEOREM 2. *If an algebroid polynomial* $f(x, y)$ *is divisible analytically by an algebroid polynomial* $\varphi(x, y)$ *of Class A in the vertex* $(c, c_1, \cdots, c_n)$ *of the latter, then* $f(x, y)$ *is algebraically divisible by* $\varphi(x, y)$ *in the point* $(y) = (c)$.

For, all the hypotheses of Theorem 1 are fulfilled throughout a definite region

$$|x| < \infty, \qquad |y_k - c_k| < \delta, \qquad k = 1, \cdots, n.$$

COROLLARY. *If* $f(x, y)$ *is also an algebroid polynomial of Class A, the quotient will be an algebroid polynomial of Class A, or else an algebroid polynomial of the* 0*th degree which does not vanish in the origin.*

In particular, if the coefficient of the highest power of $x - c$ in both f and φ is unity, the coefficient of the highest power of $x - c$ in the quotient will also be unity, or else the quotient will reduce identically to unity.

§ 6. The Algorithm of the Greatest Common Divisor.

Euclid's algorithm of the greatest common divisor, as applied to polynomials in several variables (Bôcher, l. c.), can be used with modifications now pretty obvious for algebroid polynomials. It is understood throughout that all algebroid polynomials are considered in the same point.

Let $f(x, y)$, $\varphi(x, y)$ be two algebroid polynomials of degrees $m \geqq p > 0$ respectively. By successive applications of the relation (2) of § 5 we find:

$$\text{I.} \qquad \begin{cases} P_{-1}(y) f(x, y) = Q_0(x, y)\, \varphi(x, y) & + R_1(x, y), \\ P_0(y)\, \varphi(x, y) = Q_1(x, y)\, R_1(x, y) & + R_2(x, y), \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ P_{\rho-2}(y)\, R_{\rho-2}(x, y) = Q_{\rho-1}(x, y)\, R_{\rho-1}(x, y) & + R_\rho(x, y), \\ P_{\rho-1}(y)\, R_{\rho-1}(x, y) = Q_\rho(x, y)\, R_\rho(x, y) & + R_{\rho+1}(y). \end{cases}$$

Here, $P_\lambda(y)$ and $Q_\mu(x, y)$ are algebroid polynomials, no one of which vanishes identically.

If $R_1(x, y) \equiv 0$, the system stops with the first equation. Otherwise, $R_1(x, y)$ is an algebroid polynomial of lower degree than $\varphi(x, y)$; etc. If $0 < m < p$, the same system of equations holds, only $Q_0(x, y) \equiv 0$. This amounts to interchanging f and φ.

By an *algebraic common divisor* of two algebroid polynomials $f(x, y)$ and $\varphi(x, y)$ shall be meant an algebroid polynomial of positive degree which divides them both. By a *greatest algebraic common divisor* shall be meant an algebraic common divisor of maximum degree.

THEOREM 1. *In order that two algebroid polynomials* $f(x, y)$ *and*

$\varphi(x, y)$, each of positive degree, may have an algebraic common divisor, it is necessary and sufficient that

$$R_{\rho+1}(y) \equiv 0.$$

The condition is evidently necessary. To show that it is sufficient. Here, by hypothesis, it is fulfilled. Let

$$R_\rho(x, y) = \psi(y)G(x, y),$$

where $\psi(y)$ and $G(x, y)$ are algebroid polynomials of the 0th and positive degrees respectively, and $G(x, y)$, moreover, is primitive. Then G is shown at once by the algorithm to divide both f and φ.

THEOREM 2. *If $R_{\rho+1}(y) \equiv 0$, and if $G(x, y)$ has the same meaning as above, then $G(x, y)$ is a greatest algebraic common divisor of f and φ.*

Any two greatest algebraic common divisors differ by a factor which is an algebroid polynomial of degree 0.

The proof is given at once by the algorithm, as in the case of polynomials. Every primitive algebraic common divisor of f and φ divides R_ρ and hence G.

THEOREM 3. *If two algebroid polynomials $f(x, y)$ and $\varphi(x, y)$, each of positive degree, have no algebraic common factor, then three algebroid polynomials $A(x, y)$, $B(x, y)$, $R(y)$, no one of which vanishes identically, can be found such that*

II. $$A(x, y)f(x, y) + B(x, y)\varphi(x, y) = R(y).$$

And conversely, if such a relation exists, f and φ have no algebraic common factor.

§ 7. Analogue of the Fundamental Theorem for Algebroid Polynomials.

An algebroid polynomial $f(x, y)$ shall be said to be *algebraically reducible* if it can be written as the product of two algebroid polynomials each of positive degree. If it cannot be so written, and if it is of positive degree it shall be said to be *algebraically irreducible.* Thus the algebroid polynomial

$$f(x, y) = yx$$

is algebraically irreducible.

An algebroid polynomial of the 0th degree, and the polynomial 0, are neither reducible nor irreducible.

If an algebroid polynomial of Class A is algebraically reducible, its factors are both algebroid polynomials of Class A.

The analogue of the fundamental theorem of § 4 for algebroid polynomials is as follows.

THEOREM 1. *An algebroid polynomial of positive degree, $f(x, y)$, can be*

written in one, and essentially in only one, way as the product of algebraically irreducible factors:

$$f(x, y) = [f_1(x, y)]^{\lambda_1} \cdots [f_l(x, y)]^{\lambda_l}.$$

The proof of this theorem rests on the following lemma.

LEMMA. *Let $f(x, y)$, $g(x, y)$ and $\varphi(x, y)$ be three algebroid polynomials, of which the last is primitive and algebraically irreducible; and let $f(x, y)$ not be algebraically divisible by $\varphi(x, y)$. If the product $f(x, y)\,g(x, y)$ is algebraically divisible by $\varphi(x, y)$, then $g(x, y)$ is algebraically divisible by $\varphi(x, y)$.*

The proof of the lemma is given as in the case of ordinary polynomials, either Euclid's algorithm or the relation II. being used for this purpose. Hence the above theorem is established.

Algebroid Polynomials of Class A. For these the two definitions of reducibility—algebraic reducibility and the analytic reducibility defined in § 4—come together. In other words:

THEOREM 2. *In order that an algebroid polynomial of Class A be algebraically irreducible in the point $(y) = (c)$, it is necessary and sufficient that it be analytically irreducible in its vertex $(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n)$.*

The condition is obviously sufficient. To show that it is also necessary, let $f(x, y)$ be algebraically irreducible, and suppose

$$f(x, y) = \varphi(x, y)\psi(x, y), \tag{4}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are both analytic in the vertex $(c, c_1, \cdots, c_n)$ of f and φ, and vanish there. Then

$$\varphi(x, 0) \not\equiv 0,$$

since otherwise $f(x, 0)$ would vanish identically. Hence, by Weierstrass's Theorem of Factorization $\varphi(x, y)$ can be taken as an algebroid polynomial of Class A.

From Theorem 2 of § 5 and its corollary it now follows that the other factor, $\psi(x, y)$, is also an algebroid polynomial of Class A, or else an algebroid polynomial of the 0th degree, which does not vanish in the origin. Both of these cases lead to a contradiction, and thus the theorem is established.

§ 8. Proof of the Fundamental Theorem in the General Case.

We turn now to the proof of the fundamental theorem of § 4 in the general case. Let a linear transformation be made whereby the variables $(z_1, \cdots, z_n)$ are replaced by the variables $(x, y_1, \cdots, y_\nu)$, $\nu = n - 1$, and $F(z_1, \cdots, z_n)$ goes over into a function $\mathfrak{F}(x, y_1, \cdots, y_\nu)$ which can be

treated by the theorem of factorization in its original form:

$$(z) \sim (x, y) = (0, 0);$$
$$\mathfrak{F}(x, 0, \cdots, 0) \not\equiv 0,$$
$$F(z_1, \cdots, z_n) = \mathfrak{F}(x, y_1, \cdots, y_\nu) = f(x, y)\Omega(x, y),$$

where $f(x, y)$ is an algebroid polynomial of Class A with its vertex in the origin.

By the theorem of the last paragraph $f(x, y)$ can be written as the product of a finite number of factors each analytically irreducible in the origin; and since such factors remain irreducible when a linear transformation is made, one half of the proof is thus given, namely, that $F(z_1, \cdots, z_n)$ can be written at least in one way as the product of a finite number of factors, each irreducible in the point $(z) = (a)$:

$$F(z_1, \cdots, z_n) = [F_1(z_1, \cdots, z_n)]^{\lambda_1} \cdots [F_l(z_1, \cdots, z_n)]^{\lambda_l}.$$

Suppose a second factorization of this sort were possible:

$$F(z_1, \cdots, z_n) = [\Phi_1(z_1, \cdots, z_n)]^{\mu_1} \cdots [\Phi_m(z_1, \cdots, z_n)]^{\mu_m}.$$

Then the same linear transformation as was used before will carry each function $\Phi_k(z_1, \cdots, z_n)$ into a function

$$\Psi_k(x, y_1, \cdots, y_\nu), \qquad \Psi_k(x, 0, \cdots, 0) \not\equiv 0,$$

to which Weierstrass's Theorem of Factorization is applicable. Hence

$$\begin{aligned}\mathfrak{F}(x, y) &= f(x, y)\Omega(x, y)\\ &= [f_1(x, y)]^{\lambda_1} \cdots [f_l(x, y)]^{\lambda_l}\Omega(x, y)\\ &= [\varphi_1(x, y)]^{\mu_1} \cdots [\varphi_q(x, y)]^{\mu_q}\,\mathrm{X}(x, y)\end{aligned}$$

and

$$f(x, y) = [\varphi_1(x, y)]^{\mu_1} \cdots [\varphi_q(x, y)]^{\mu_q}\frac{\mathrm{X}(x, y)}{\Omega(x, y)}.$$

Thus the algebroid polynomial of Class A, $f(x, y)$, is seen to be divisible analytically at its vertex by an algebroid polynomial of Class A. By Theorem 2 of § 5 and its corollary the quotient is also an algebroid polynomial, and since it does not vanish in the origin, it does not depend on x.

Finally, then, by the theorems of § 7 the factors $f_k(x, y)$ and $\varphi_k(x, y)$ are respectively equivalent to each other and occur with like multiplicities. This proves the theorem.

§ 9. A General Theorem Relating to Divisibility.

A theorem of frequent application in algebraic geometry is the following. If two algebraic curves have ever so restricted an arc in common, they have a whole irreducible algebraic curve in common.

The theorem can be generalized for arbitrary analytic hypersurfaces as follows.

THEOREM.* *If $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ are two functions each analytic in the point $(z) = (a)$ and vanishing there, neither function vanishing identically, and if F vanishes for every root of Φ which lies in the neighborhood of (a), then each irreducible factor of Φ in (a) is also a factor of F in (a).*

It is evidently sufficient to prove the theorem for the case that Φ is irreducible in (a). Let a linear transformation be made whereby F and Φ go over into functions to which the theorem of factorization in its first form is applicable:

$$F(z_1, \cdots, z_n) = \mathfrak{F}(x, y_1, \cdots, y_{n-1}), \qquad \Phi(z_1, \cdots, z_n) = \Psi(x, y_1, \cdots, y_{n-1}),$$

$$\mathfrak{F}(x, 0, \cdots, 0) \not\equiv 0, \qquad \Psi(x, 0, \cdots, 0) \not\equiv 0,$$

$$\mathfrak{F}(x, y) = f(x, y)\Omega(x, y); \qquad \Psi(x, y) = \psi(x, y)\mathrm{X}(x, y).$$

Here, $\psi(x, y)$ is an irreducible algebroid polynomial of Class A. If the algebroid polynomial $f(x, y)$ does not admit $\psi(x, y)$ as a factor, then f is algebraically prime to ψ, and by II, § 6, we have:

$$A(x, y)f(x, y) + B(x, y)\psi(x, y) = R(y).$$

Let (y') be a point in the neighborhood of the origin, for which

$$R(y') \neq 0,$$

and let x' be a root of $\psi(x, y')$. Then the left-hand side of this equation vanishes at the point (x', y'), while the right-hand side does not. From this contradiction follows the truth of the theorem.

Proof of the Last Theorem of § 4. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ be relatively prime in the point $(z) = (a)$, and let them be transformed as above into functions $\mathfrak{F}(x, y)$ and $\Psi(x, y)$. Then there exists a relation of the form

$$A(x, y)f(x, y) + B(x, y)\psi(x, y) = R(y),$$

where $R(y)$ does not vanish identically. If, now, (z^0) be any point of the neighborhood of (a) and (x^0, y^0) the corresponding point, this same relation holds in the neighborhood of the latter point, and hence f and ψ are relatively prime at (x^0, y^0). This shows that F and Φ are likewise relatively prime at (z^0), *q.e.d.*

HARVARD UNIVERSITY,
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* Hölder, Mathematisch-naturwissenschaftliche Mitteilungen, I (1884); Tübingen, Fues. Study, Ternäre Formen, p. 202.